THE STRUCTURE OF ZHU'S ALGEBRAS FOR CERTAIN $\mathcal{W}$-ALGEBRAS

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ABSTRACT. We introduce a new approach that allows us to determine the structure of Zhu's algebra for certain vertex operator (super)algebras which admit horizontal $\mathbb{Z}$-grading. By using this method and an earlier description of Zhu's algebra for the singlet $\mathcal{W}$-algebra, we completely describe the structure of Zhu's algebra for the triplet vertex algebra $\mathcal{W}(p)$. As a consequence, we prove that Zhu's algebra $A(\mathcal{W}(p))$ and the related Poisson algebra $\mathcal{P}(\mathcal{W}(p))$ have the same dimension. We also completely describe Zhu's algebras for the $N=1$ triplet vertex operator superalgebra $\mathcal{SW}(m)$. Moreover, we obtain similar results for the $c=0$ triplet vertex algebra $\boldsymbol{\mathcal{W}}_{2,3}$, important in logarithmic conformal field theory. Because our approach is "internal" we had to employ several constant term identities for purposes of getting right upper bounds on $\dim(A(V))$.

This work is, in a way, a continuation of the results published in [4].

1. INTRODUCTION

In this work we address two important algebraic objects that can be associated to any conformal vertex (super)algebra V, both introduced in a seminal paper by Zhu [23]:

(i) Zhu's associative algebra $A(V)$, and
(ii) commutative Poisson algebra $\mathcal{P}(V) = V/C_2(V)$.

The Zhu's algebra $A(V)$ is instrumental in representation theory of vertex algebra and has been a subject of numerous papers. On the other hand, $\mathcal{P}(V)$ is primarily used for purposes of modular invariance of graded dimensions [23]. The two algebras are of course closely related; we always have a natural surjective map from $\mathcal{P}(V)$ to $\mathrm{gr}A(V)$ (the associated graded algebra of $A(V)$), giving

$$\dim(\mathcal{P}(V)) \geq \dim(A(V)), \tag{1}$$

at least if $\mathcal{P}(V)$ is finite-dimensional (i.e. V is C_2-cofinite).

Fairly recently, Gaberdiel and Gannon [15] have initiated a thorough study of possible relationships between $\mathcal{P}(V)$ and $A(V)$, by focusing primarily to rational vertex algebras of affine type. Although they have observed that for many familiar examples - such as Virasoro minimal models - these two algebras will have the same dimension (and thus the above map will be an isomorphism), there are many instances for which this is false (take for instance V to be the level one affine vertex algebra associated to Lie algebra of type E_8). The main observation is that the discrepancy between two algebras is somewhat "controlled" by twisted V-modules. They also provided a conjecture for the equality of dimensions for certain vertex algebras associated to representations of affine algebras at positive integral levels. Some of their conjectures have been recently settled in [9] and [10].

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In view of (1) we can also contemplate whether $\mathcal{P}(V)$ and $\mathrm{gr}A(V)$ are isomorphic for V being C_2-cofinite, but not necessarily rational. Such vertex algebras have recently attracted a lot of attention in connection with logarithmic conformal field theory [17],[20]. But even for the well-known triplet vertex algebra $\mathcal{W}(p)$ (cf. [11], [14], [4]), the two algebras have not been determined in full yet for all p (see however [4]) . The same issue can be also addressed for C_2-cofinite vertex operators superalgebras. Again, very little is known apart from vertex algebras associated to $N=1$ minimal models (cf. [1] and [19]).

The aims of this paper include: (a) We present a new approach for determining Zhu's algebra for vertex algebra which admit horizontal $\mathbb{Z}$-grading (and some additional properties). This method is successfully applied to triplet vertex algebras. (b) We settle several conjectures from our previous work needed for better description of $\mathcal{P}(V)$ in the case of the triplet algebras $\mathcal{W}(p)$. (c) Finally, we show how to extend our results to vertex operator superalgebras.

So let us briefly outline the main results.

We closely follow [4] and [6], and as before let $\mathcal{W}(p)$ be the triplet vertex operator algebra and $\mathcal{SW}(m)$ be the $N=1$ supertriplet algebra. From the structural results about Zhu's algebras from [4], [5] and [6], we have that for the complete description of Zhu's algebras $A(\mathcal{W}(p))$, $A(\mathcal{SW}(m))$ and $A_\sigma(\mathcal{SW}(m))$ one has to describe the center of these algebras. Let us describe our approach in the case of triplet vertex algebra. Let $\overline{M(1)} \subset \mathcal{W}(p)$ be the singlet vertex operator algebra (cf. [2], [3]). We investigate a natural homomorphism of Zhu's algebras $\Phi : A(\overline{M(1)}) \rightarrow A(\mathcal{W}(p))$ and identify the center of $A(\mathcal{W}(p))$ as a subalgebra of $A(\overline{M(1)})/\ \mathrm{Ker}(\Phi)$. We prove that the kernel $\mathrm{Ker}(\Phi)$ is a principal ideal in Zhu's algebra $A(\overline{M(1)})$, and show that the dimension of $A(\overline{M(1)})/\ \mathrm{Ker}(\Phi)$ is $4p-1$. This easily gives the description of the center and proves that the dimension of $A(\mathcal{W}(p))$ is $6p-1$. We also prove that:

Theorem 1.1. *For $p \geq 2$,*
$$\dim A(\mathcal{W}(p)) = \dim \mathcal{P}(\mathcal{W}(p)).$$

For vertex operator superalgebras (with suitable grading) the situation is a bit different because there are four distinct algebras of interest here: the usual untwisted Zhu's algebra $A(V)$, its σ-twisted counterpart $A_\sigma(V)$, the Poisson superalgebra $\mathcal{P}(V)$ and its even subalgebra $\mathcal{P}_0(V)$. It can be shown that there is again a surjective map from $\mathcal{P}_0(V)$ to $\mathrm{gr}A(V)$, giving an upper bound on $\dim(A(V))$. But more important object of consideration turns out to be $A_\sigma(V)$. By applying similar approach we show that the dimension of $A(\mathcal{SW}(m))$ is $6m+1$ and the dimension of $A_\sigma(\mathcal{SW}(m))$ is $12m+8$. In this way we present a positive answer on conjectures from papers [5]-[6].

Theorem 1.2. *For all $m \in \mathbb{N}$,*
$$\dim A_\sigma(\mathcal{SW}(m)) = \dim \mathcal{P}(\mathcal{SW}(m)).$$

This equality is known to hold for $N=1$ vertex superalgebras associated to $N=1$ minimal models (cf. [19]).

Interestingly, the natural homomorphism from $\mathcal{P}_0(\mathcal{SW}(m))$ onto $\mathrm{gr}(A(\mathcal{SW}(m)))$ yields a non-trivial kernel. Namely, we have

Corollary 1.3. *For all $m \in \mathbb{N}$,*
$$\dim A(\mathcal{SW}(m)) < \dim \mathcal{P}_0(\mathcal{SW}(m)). \quad (2)$$

Our methods can be of course applied for "logarithmic extension" of (p, q) Virasoro minimal models (cf. [7], [12], [16]). Thus in Section 5 we completely describe the Zhu algebra for the vertex algebra $\boldsymbol{\mathcal{W}}_{2,3}$ of central charge zero. As a consequence, we prove the following result, predicted in the physics literature

Corollary 1.4. *The vertex algebra $\boldsymbol{\mathcal{W}}_{2,3}$ admits a logarithmic module of $L(0)$ nilpotent rank 3 (here $L(0)$ denotes a generator of the Virasoro algebra).*

It is important to observe that our approach is "internal" in a sense that only properties of the vertex algebra in question have been used to determine Zhu's algebra. We do not use any information about the structure or existence of logarithmic representations (which is in general nontrivial), nor modular invariance [4], [20]. Instead, the existence of logarithmic modules is obtained from complete structure of the Zhu algebra. In this way we give additional evidence for the correspondence between the category of modules for triplet vertex algebras and the category of modules for certain quantum groups which are Kazhdan-Lusztig duals of triplet vertex algebras (cf. [11], [13]).

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2. Main definitions

The starting point for this paper is to recall the definition of Zhu's algebra for vertex operator (super)algebras following [18], [23].

Let $(V = V^{\bar{0}} \oplus V^{\bar{1}}, Y, \mathbf{1}, \omega)$ be a vertex operator superalgebra. We shall always assume that

$$V^{\bar{0}} = \coprod_{n \in \mathbb{Z}_{\geq 0}} V(n), \quad V^{\bar{1}} = \coprod_{n \in \frac{1}{2} + \mathbb{Z}_{\geq 0}} V(n)$$

$$\text{where } \; V(n) = \{a \in V \mid L(0)a = nv\}.$$

For $a \in V(n)$, we shall write $\mathrm{wt}(a) = n$, or $\deg(a) = n$. As usual, vertex operator associated to $a \in V$ is denoted by $Y(a, x)$, with the mode expansion

$$Y(a, x) = \sum_{n \in \mathbb{Z}} a_n x^{-n-1}.$$

We define two bilinear maps $* : V \times V \to V$, $\circ : V \times V \to V$ as follows: for homogeneous $a, b \in V$, let

$$a * b = \begin{cases} \mathrm{Res}_x Y(a, x) \frac{(1+x)^{\deg(a)}}{x} b & \text{if } a, b \in V^{\bar{0}} \\ 0 & \text{if } a \text{ or } b \in V^{\bar{1}} \end{cases} \tag{3}$$

$$a \circ b = \begin{cases} \mathrm{Res}_x Y(a, x) \frac{(1+x)^{\deg(a)}}{x^2} b & \text{if } a \in V^{\bar{0}} \\ \mathrm{Res}_x Y(a, x) \frac{(1+x)^{\deg(a) - \frac{1}{2}}}{x} b & \text{if } a \in V^{\bar{1}} \end{cases} \tag{4}$$

Next, we extend $*$ and $\circ$ to $V \otimes V$ linearly, and denote by $O(V) \subset V$ the linear span of elements of the form $a \circ b$, and by $A(V)$ the quotient space $V/O(V)$. The space $A(V)$ has a unitary

associative algebra structure, with the multiplication induced by $*$. Algebra $A(V)$ is called the Zhu's algebra of V. The image of $v \in V$, under the natural map $V \mapsto A(V)$ will be denoted by $[v]$.

For a homogeneous $a \in V$ we define

$$o(a) = a_{\mathrm{wt}(a)-1}.$$

In the case when $V^{\bar{0}} = V$, V is a vertex operator algebra and we get the usual definition of Zhu's algebra for vertex operator algebras.

With V as above, we let

$$C_2(V) = \langle a_{-2}b : a, b \in V \rangle \ \text{ and } \ \mathcal{P}(V) = V/C_2(V).$$

The quotient space $\mathcal{P}(V)$ has an algebraic structure of a commutative Poisson algebra [23]. Explicitly, if we denote by $\bar{a}$ the image of a under the natural map $V \mapsto \mathcal{P}(V)$ the Poisson bracket is given by $\{\overline{a}, \overline{b}\} = \overline{a_0 b}$ and commutative product $\overline{a} \cdot \overline{b} = \overline{a_{-1} b}$. If V is a vertex superalgebra, clearly $\mathcal{P}(V)$ is $\mathbb{Z}_2$-graded. Its even part will be denoted by $\mathcal{P}_0(V)$. From the given definitions it is not hard to construct an increasing filtration of $A(V)$ such that $\mathrm{gr}A(V)$ maps onto $\mathcal{P}(V)$.

3. A homomorphism of Zhu's algebras

Assume that V is a vertex operator superalgebra which admits the (horizontal) $\mathbb{Z}$–gradation:

$$V = \bigoplus_{\ell \in \mathbb{Z}} V_\ell, \quad V_{\ell_1} \cdot V_{\ell_2} \subset V_{\ell_1 + \ell_2},$$

where

$$V_k \cdot V_\ell := \mathrm{span}_{\mathbb{C}} \{u_n v : u \in V_k, v \in V_\ell, n \in \mathbb{Z}\}.$$

In addition, assume there is $G \in \mathrm{End}(V)$ such that:

(5) $$G \quad \text{is a derivation on V},$$

(6) $$G(V_\ell) \subset V_{\ell+1},$$

(7) $$\omega \in V_0, \quad [G, Y(\omega, z)] = 0,$$

(8) $$G|V_\ell \quad \text{is injective for } \ \ell < 0,$$

(9) $$G|V_\ell \quad \text{is surjective for } \ \ell \geq 0.$$

Clearly, V_0 is a subalgebra of V.

We shall consider Zhu's algebras

$$A(V_0) = V_0/O(V_0), \quad A(V) = V/O(V).$$

Let

$$O(V)_0 = O(V) \cap V_0 = \mathrm{span}_{\mathbb{C}} \{v \circ w \mid v \in V_\ell, \ w \in V_{-\ell}, \ \ell \in \mathbb{Z}\}.$$

Consider the following homomorphism of Zhu's algebras

$$\begin{aligned} \Phi : \quad & A(V_0) \to A(V) \\ & v + O(V_0) \mapsto v + O(V), \quad v \in V_0. \end{aligned}$$

Let
$$A_0(V) = \mathrm{Im}(\Phi) = \frac{V_0}{O(V)_0}.$$
Clearly,
$$A_0(V) \cong A(V_0)/\mathrm{Ker}(\Phi).$$
We are interested in $\mathrm{Ker}(\Phi)$. First we notice that
$$\mathrm{Ker}(\Phi) = \{u + O(V_0) \in A(V_0) \mid u \in O(V)_0\}.$$

Lemma 3.1. *We have*
$$O(V)_0 \subset G(V_{-1}) + O(V_0).$$

Proof. We have to prove that for every $\ell \in \mathbb{Z}$:

$$v \in V_\ell, \quad w \in V_{-\ell} \quad \Longrightarrow \; v \circ w \in G(V_{-1}) + O(V_0). \tag{10}$$

We consider the case when $\ell \in \mathbb{Z}_{\geq 0}$. The case $\ell < 0$ can be proved analogously.

We prove the claim (10) by induction on $\ell \in \mathbb{Z}_{\geq 0}$. For $\ell = 0$ the claim holds.

Assume that $\ell = 1$, $v \in V_1$, $w \in V_{-1}$.

Take $v' \in V_0$ such that $v = Gv'$. Since $v' \circ w \in V_{-1}$ and $v' \circ Gw \in O(V_0)$, we have
$$v \circ w = G(v' \circ w) - v' \circ Gw \in G(V_{-1}) + O(V_0).$$

Assume now that the claim holds for $\ell > 0$. Let
$$v \in V_{\ell+1}, \quad w \in V_{-\ell-1}.$$

Take $v' \in V_\ell$ such that $Gv' = v$. By using induction hypothesis and the fact that $v' \circ w \in V_{-1}$, we get
$$v \circ w = G(v' \circ w) - v' \circ Gw \in G(V_{-1}) + O(V_0).$$
The proof follows. □

Lemma 3.1 implies the following result:

Proposition 3.2. *Let* $[G(V_{-1})] = \{[Gv],\ v \in V_{-1}\}$. *Then*
$$\mathit{Ker}(\Phi) \subset [G(V_{-1})].$$

4. STRUCTURE OF ZHU'S ALGEBRA $A(\mathcal{W}(p))$

In this section we shall describe the structure of Zhu's algebra of the triplet vertex algebra $\mathcal{W}(p)$. We use several structural results on triplet and singlet vertex algebra obtained in [2], [3] and [4], which we recall briefly.

Let $p \in \mathbb{Z}$ such that $p \geq 2$. As usual, V_L will denote the lattice vertex algebra associated to the positive definite even lattice
$$L = \mathbb{Z}\alpha, \quad \langle \alpha, \alpha \rangle = 2p.$$
Let Y be the associated vertex operator. For details of the construction see, for instance, [4].

The triplet vertex algebra $\mathcal{W}(p)$, of *type* $(2, 2p-1, 2p-1, 2p-1)$, is a vertex subalgebra of V_L generated by the conformal vector
$$\omega = \frac{1}{4p}\alpha(-1)^2 + \frac{p-1}{2p}\alpha(-2),$$

and the *primary* vectors

$$F = e^{-\alpha},\ H = QF,\ E = Q^2 e^{-\alpha},$$

where $Q = e_0^{\alpha} = \mathrm{Res}_z Y(e^{\alpha}, z)$ is a so-called *screening* operator. The operator Q acts horizontally (preserving conformal weight) so that

$$\deg(X) = 2p-1, \quad X \in \{E, F, H\}.$$

There is another useful description of $\mathcal{W}(p)$. As a module for the Virasoro algebra, V_L is not completely reducible. But it has a semisimple submodule which is isomorphic to the triplet vertex algebra. More precisely,

$$\begin{aligned} \mathcal{W}(p) &= \mathrm{soc}_{Vir}(V_L) \\ &= \bigoplus_{n=0}^{\infty} \bigoplus_{j=0}^{2n} U(Vir).Q^j e^{-n\alpha}. \end{aligned} \tag{11}$$

The vertex subalgebra of $\mathcal{W}(p)$ generated by ω and H is called singlet vertex algebra and will be denoted by $\overline{M(1)}$. Clearly, $\overline{M(1)}$ is a subalgebra of the Heisenberg vertex algebra $M(1)$.

For $i \in \mathbb{Z}$ we set

$$h_{i,1} = \frac{(p-i)^2 - (p-1)^2}{4p}.$$

The triplet $\mathcal{W}(p)$ is known to be C_2-cofinite but irrational [4]. Moreover, $\mathcal{W}(p)$ has precisely $2p$ inequivalent irreducible modules which are usually denoted by

$$\Lambda(1), \dots, \Lambda(p), \Pi(1), \dots, \Pi(p).$$

For $1 \le i \le p$, the top component of $\Lambda(i)$ is 1-dimensional and has conformal weight $h_{i,1}$, and the top component of $\Pi(i)$ is 2–dimensional with conformal weight $h_{3p-i,1}$.

Theorem 4.1. [2] *Zhu's associative algebra $A(\overline{M(1)})$ is isomorphic to the commutative algebra $\mathbb{C}[x,y]/\langle P(x,y)\rangle$, where $\langle P(x,y)\rangle$ is the principal ideal generated by*

$$P(x,y) = y^2 - \frac{(4p)^{2p-1}}{(2p-1)!^2}\,(x + \frac{(p-1)^2}{4p}) \prod_{i=0}^{p-2} \left(x + \frac{i}{4p}(2p-2-i) \right)^2. \tag{12}$$

(Here x and y correspond to $[\omega]$ and $[H]$).

Remark 1. *Description in Theorem 4.1 shows that the subalgebra P of $A(\overline{M(1)})$ generated by $[\omega]$ is isomorphic to $\mathbb{C}[x]$ and that 1 and $[H]$ are algebraically independent over $\mathbb{C}[x]$, meaning that if*

$$A([\omega]) + B([\omega]) * [H] = 0 \ \text{ in } \ A(\overline{M(1)}),$$

for some polynomials A and B then $A(x) = B(x) = 0$.

Recall that as a module over Virasoro algebra $\mathcal{W}(p)$ is generated by singular vectors

$$\{Q^j e^{-n\alpha},\ n \in \mathbb{Z}_{\ge 0}, 0 \le j \le 2n\},$$

and $\overline{M(1)}$ is generated by singular vectors

$$\{Q^n e^{-n\alpha},\ n \in \mathbb{Z}_{\ge 0}\}.$$

For every $\ell \in \mathbb{Z}_{\geq 0}$, we define

$$\mathcal{W}(p)_{-\ell} = \overline{M(1)}.e^{-\ell\alpha} = \bigoplus_{n=0}^{\infty} U(Vir).Q^n e^{-(n+\ell)\alpha},$$

$$\mathcal{W}(p)_{\ell} = \overline{M(1)}.Q^{2\ell} e^{-\ell\alpha} = \bigoplus_{n=0}^{\infty} U(Vir).Q^{n+2\ell} e^{-(n+\ell)\alpha}.$$

Then we have yet another description of $\mathcal{W}(p)$.

Proposition 4.2. *For every $\ell \in \mathbb{Z}$, $\mathcal{W}(p)_\ell$ is an irreducible $\overline{M(1)}$–module and*

$$\mathcal{W}(p) = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{W}(p)_\ell.$$

Moreover, for $v \in \mathcal{W}(p)_{\ell_1}$, $w \in \mathcal{W}(p)_{\ell_2}$, we have

$$Y(v,z)w \in \mathcal{W}(p)_{\ell_1+\ell_2}((z)).$$

Operator $G = Q$ satisfies conditions (5)-(9).

Proof. Irreducibility has been established in [3], and the decomposition follows from description given in [4]. □

Lemma 4.3. *We have*

$$O(\mathcal{W}(p))_0 \subset \bigoplus_{n=1}^{\infty} U(Vir).Q^n e^{-n\alpha} + O(\overline{M(1)}).$$

Proof. First we notice that

$$Q(\mathcal{W}(p)_{-1}) = \bigoplus_{n=1}^{\infty} U(Vir).Q^n e^{-n\alpha}.$$

Now assertion follows from Lemma 3.1. □

Lemma 4.4. *Assume $n \geq 1$. Then in Zhu's algebra $A(\overline{M(1)})$ we have*

$$[Q^n e^{-n\alpha}] = A([\omega]) * [H] + B([\omega]) * f_p([\omega]),$$

where

$$f_p(x) = \prod_{i=1}^{3p-1} (x - h_{i,1}),$$

and $A, B \in \mathbb{C}[x]$.

Proof. Let $Y(H,z) = \sum_{j \in \mathbb{Z}} H_j z^{-j-1}$. The results from [2] and [3] imply that

$$H_j Q^k e^{-k\alpha} \in U(Vir) Q^{k+1} e^{-(k+1)\alpha} \oplus U(Vir) Q^{k-1} e^{-(k-1)\alpha},$$

where $j \in \mathbb{Z}$ and $k \geq 1$. Moreover there exists $j_0 \leq -2$ such that

$$H_{j_0} Q^k e^{-k\alpha} = C Q^{k+1} e^{-(k+1)\alpha} + f(\omega) Q^{k-1} e^{-(k-1)\alpha} \quad (C \neq 0,\ f(\omega) \in U(Vir)).$$

This easily implies that in $A(\overline{M(1)})$

$$[Q^{k+1} e^{-(k+1)\alpha}] = \overline{f}([\omega]) * [Q^{k-1} e^{-(k-1)\alpha}]$$

for certain $\overline{f} \in \mathbb{C}[x]$. By induction, we now have that

$$[Q^n e^{-n\alpha}] = A([\omega]) * [H] + B([\omega]) * [Q^2 e^{-2\alpha}].$$

From [4] we have that $[Q^2 e^{-2\alpha}] = D f_p([\omega])$, $D \neq 0$. The proof follows. □

Now we shall use results from Section 3 and obtain the following important result.

Theorem 4.5. *We have*

$$\begin{aligned} Ker(\Phi) &= A(\overline{M(1)}).(p([\omega]) * [H]) \\ &\cong span_{\mathbb{C}}\{A([\omega]) * p([\omega]) * [H] + B([\omega]) * f_p([\omega]), \quad A, B \in \mathbb{C}[x]\}, \end{aligned}$$

where

$$p(x) = \prod_{i=2p}^{3p-1} (x - h_{i,1}), \quad f_p(x) = \prod_{i=1}^{3p-1} (x - h_{i,1}).$$

Proof. By using Proposition 3.2 and Lemma 4.4 we conclude

$$\mathrm{Ker}(\Phi) \subset \{A([\omega]) * [H] + B([\omega]) * f_p([\omega]), \quad A, B \in \mathbb{C}[x]\}.$$

Let $u \in \mathrm{Ker}(\Phi)$. Then

$$u = A([\omega]) * [H] + B([\omega]) * f_p([\omega])$$

for some $A, B \in \mathbb{C}[x]$.

Since $\mathrm{Ker}(\Phi)$ is an ideal in $A(\overline{M(1)})$ we have

$$[H] * u = \tilde{A}([\omega]) * [H] + \tilde{B}([\omega]) * f_p([\omega]),$$

for some $\tilde{A}, \tilde{B} \in \mathbb{C}[x]$. On the other hand

$$[H] * u = A([\omega]) * [H]^2 + B([\omega]) * f_p([\omega]) * [H] \in \mathrm{Ker}(\Phi).$$

Because of

$$A([\omega]) * [H]^2 = A([\omega]) * g([\omega]), \quad g(x) = C_p \prod_{i=1}^{2p-1} (x - h_{i,1}),$$

and Remark 1 we conclude that $\tilde{B}(x) f_p(x) = \tilde{B}(x) p(x) g(x) = A(x) g(x)$. Therefore, $p(x) | A(x)$. In this way we have proved that

$$\mathrm{Ker}(\Phi) \subseteq \{A_1([\omega]) * p([\omega]) * [H] + B_1([\omega]) * f_p([\omega]), \quad A_1, B_1 \in \mathbb{C}[x]\}.$$

Now the results from [4] imply the relation

$$\Phi\Big(p([\omega]) * [H]\Big) = \Phi\Big(f_p([\omega])\Big) = 0,$$

which proves the opposite inclusion. The proof follows. □

Let

$$A_0(\mathcal{W}(p)) = \mathrm{Im}(\Phi) \cong A(\overline{M(1)})/\mathrm{Ker}(\Phi).$$

The results from [4] (cf. Theorem 5.1 of [4]) imply that

$$A(\mathcal{W}(p)) = A_{-1}(\mathcal{W}(p)) \oplus A_0(\mathcal{W}(p)) \oplus A_1(\mathcal{W}(p)),$$

where

$$A_{-1}(\mathcal{W}(p)) = A_0(\mathcal{W}(p)).[F], \quad A_1(\mathcal{W}(p)) = A_0(\mathcal{W}(p)).[E],$$

and

$$\dim A_{\pm 1}(\mathcal{W}(p)) = p, \quad \dim A_0(\mathcal{W}(p)) \leq 4p-1.$$

Now Theorem 4.5 implies that $\dim A_0(\mathcal{W}(p)) = 4p-1$. In this way we have proved the following theorem:

Theorem 4.6. *For $p \geq 2$, we have*

$$\dim A(\mathcal{W}(p)) = 6p-1.$$

In [4], we have proved that $\dim A(\mathcal{W}(p)) \leq 6p-1$ and that $\dim A(\mathcal{W}(p)) = 6p-1$ if and only if $A(\mathcal{W}(p))$ contains 2-dimensional ideals $\mathbb{I}_{h_{i,1}}$ parameterized by conformal weights $h_{i,1}$ (see Section 5 of [4]). These ideals give $(p-1)$ indecomposable 2-dimensional representations. Then our Theorem 4.6 implies that $A(\mathcal{W}(p))$ has these indecomposable representations. By using Zhu's correspondence we obtain the following result.

Corollary 4.7. *For every $1 \leq i \leq p-1$, there exits the logarithmic, self-dual, $\mathbb{Z}_{\geq 0}$–graded $\mathcal{W}(p)$–module $\mathcal{P}_i^+$ such that the top component $\mathcal{P}_i^+(0)$ is two-dimensional and $L(0)$ acts on $\mathcal{P}_i^+(0)$ as*

$$\left(\begin{array}{cc} h_{i,1} & 1 \\ 0 & h_{i,1} \end{array} \right).$$

Remark 2. *The existence of logarithmic modules (in the case p is a prime number) was proved in* [4] *by using modular invariance. Here we have presented a proof which use only theory of Zhu's algebras. We show that existence of logarithmic modules can detected directly from the internal structure of Zhu's algebra.*

Theorem 4.8. *For every $p \geq 2$*

$$\dim(\mathcal{P}(\mathcal{W}(p)) \leq 6p-1.$$

Proof. The description of $C_2(\mathcal{W}(p))$ from [4] gives that $\mathcal{P}(\mathcal{W}(p))$ is generated by

$$\overline{\omega}, \overline{H}, \overline{E}, \overline{F},$$

(here $\overline{a}$ denotes the image of a in $\mathcal{P}(\mathcal{W}(p))$), and that the following relations hold:

$$\overline{\omega}^{3p-1} = \overline{E}^2 = \overline{F}^2 = \overline{H} \cdot \overline{E} = \overline{H} \cdot \overline{F} = 0, \tag{13}$$

$$\overline{H}^2 = -\overline{E} \cdot \overline{F} = \nu \overline{\omega}^{2p-1} \quad (\nu \neq 0). \tag{14}$$

By using (11) and the fact that $\mathrm{wt}(H_{-2}F) = 4p-1$ we get

$$H_{-2}F = \nu L(-2)^p F + v_1, \quad v_1 \in C_2(\mathcal{W}(p)), \quad \nu \in \mathbb{C}.$$

Applying the screening operator Q and using the fact that $C_2(\mathcal{W}(p))$ is Q–invariant, we get

$$E_{-2}F = \nu L(-2)^p H + v_2, \quad E_{-2}H = \nu L(-2)^p E + v_3$$

where $v_2, v_3 \in C_2(\mathcal{W}(p))$.

Another important ingredient is Theorem 9.1 below, which implies that in $A(\overline{M(1)})$ we have the following relation

$$[E \circ F] = \nu_1 p([\omega]) * [H], \qquad (\nu_1 \neq 0).$$

But this relation in Zhu's algebra easily gives that $\nu = \nu_1 \neq 0$.

Therefore we have proved

$$\overline{\omega}^p \overline{H} = \overline{\omega}^p \overline{E} = \overline{\omega}^p \overline{F} = 0. \tag{15}$$

Finally, relations (13) – (15) prove the inequality

$$\dim(\mathcal{P}(\mathcal{W}(p)) \leq 6p - 1.$$

□

Combined together

Corollary 4.9. *We have*

$$\dim A(\mathcal{W}(p)) = \dim(\mathcal{P}(\mathcal{W}(p))) = 6p - 1.$$

5. On Zhu's algebra $A(\boldsymbol{\mathcal{W}}_{2,3})$

In this section we shall consider the triplet vertex algebra $\boldsymbol{\mathcal{W}}_{2,p}$ from [7] and [12].

Assume p is an odd natural number, $p \geq 3$, and let

$$L = \mathbb{Z}\alpha, \quad \langle \alpha, \alpha \rangle = p.$$

Let V_L be the associated vertex superalgebra. We recall that V_L is generated by vectors e^{α} and $e^{-\alpha}$. As usual, let

$$Y(e^{\beta}, z) = \sum_{i \in \mathbb{Z}} e^{\beta}_i z^{-i-1}, \quad \beta \in L$$

Define the Virasoro vector

$$\omega = \frac{1}{2p}(\alpha(-1)^2 + (p-2)\alpha(-2))\mathbf{1}$$

and the (screening) operators $Q = e^{\alpha}_0$ and

$$G = \sum_{i=1}^{\infty} \frac{1}{i} e^{\alpha}_{-i} e^{\alpha}_i.$$

As shown in [7] the triplet vertex algebra $\boldsymbol{\mathcal{W}}_{2,p}$ can be realized as a subalgebra of V_L generated by ω and primary vectors

$$F = Qe^{-3\alpha}, \quad H = GF, \quad E = G^2 F.$$

The results from [7] give that the results from Section 3 can be applied on $\boldsymbol{\mathcal{W}}_{2,p}$. In particular $\boldsymbol{\mathcal{W}}_{2,p}$ is $\mathbb{Z}$–graded and charge zero component is the singlet vertex algebra $\overline{M(1)}$ generated by ω and H. We also have homomorphism $\Phi : A(\overline{M(1)}) \to A(\boldsymbol{\mathcal{W}}_{2,p})$.

We shall now consider the case $p = 3$. By using analogous approach as in the case of $\mathcal{W}(p)$ we prove the following result.

Proposition 5.1. *Ker*(Φ) *is contained in the following ideal in* $A(\overline{M(1)})$

$$A(\overline{M(1)}).\{p([\omega]) * [H], f_{2,3}([\omega])\} = \{A([\omega]) * p([\omega]) * [H] + B([\omega]) * f_{2,3}([\omega]), \ \ A, B \in \mathbb{C}[x]\},$$

where

$$\begin{aligned} p(x) = \quad & (x-5)(x-7)(x-\tfrac{10}{3})(x-\tfrac{33}{8})(x-\tfrac{21}{8})(x-\tfrac{35}{24}) \\ f_{2,3}(x) = \quad & x^3\left((x-1)(x-2)(x-\tfrac{1}{8})(x-\tfrac{5}{8})(x-\tfrac{1}{3})\right)^2 \\ & (x-5)(x-7)(x-\tfrac{10}{3})(x+\tfrac{1}{24})(x-\tfrac{33}{8})(x-\tfrac{21}{8})(x-\tfrac{35}{24}). \end{aligned}$$

Corollary 5.2. *The center of Zhu's algebra $A(\boldsymbol{\mathcal{W}}_{2,3})$ is 20–dimensional and it is isomorphic to*

$$\mathbb{C}[x]/\langle f_{2,3}(x)\rangle.$$

Proof. By using results from [7] one can easily see that the center of $A(\boldsymbol{\mathcal{W}}_{2,3})$ is isomorphic to the subalgebra generated by $[\omega]$. By using Proposition 5.1 we see that

$$f\in\mathbb{C}[x],\ f([\omega])\in\mathrm{Ker}(\Phi) \quad \Longrightarrow \ f_{2,3}|f.$$

Since $f_{2,3}([\omega])\in\mathrm{Ker}(\Phi)$, we prove the assertion. $\square$

Remark 3. *This result implies that $A(\boldsymbol{\mathcal{W}}_{2,3})$ has 2–dimensional indecomposable modules $U_h^{(2)}$ on which $[\omega]$ acts (in some basis) as*

$$\begin{pmatrix} h & 1 \\ 0 & h \end{pmatrix}$$

where $h\in\{0,1,2,1/8,5/8,1/3\}$, and 3–dimensional indecomposable module $U_0^{(3)}$ on which $[\omega]$ acts as

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Other zeros of the polynomial $f_{2,3}$ do not give rise to indecomposable modules.

By applying the theory of Zhu's algebras, we get the existence of logarithmic, $\mathbb{Z}_{\geq 0}$–graded $\boldsymbol{\mathcal{W}}_{2,3}$–modules $R_h^{(2)}$ and $R_0^{(3)}$ whose top components are isomorphic to $U_h^{(2)}$ and $U_0^{(3)}$ respectively. These modules appeared in the fusion rules analysis in [16].

By using representation theory of the vertex operator algebra $\boldsymbol{\mathcal{W}}_{2,3}$ we can conclude that

$$0=[F\circ E]=g([\omega])*p([\omega])*[H], \tag{16}$$

for certain polynomial g of degree 2. We can determine polynomial g by evaluating (16) on top components of $\overline{M(1)}$–modules.

By using direct calculation (and Mathematica) we get:

$$g(x)=\nu(\frac{62128128}{14003665}x^2-\frac{918683648}{14003665}x+\frac{5767168}{215441}) \quad (\nu\neq 0).$$

Moreover, g is relatively prime with $f_{2,3}$. Therefore, $\Phi(g([\omega]))$ is invertible in $A(\boldsymbol{\mathcal{W}}_{2,3})$ and we have that in $p([\omega])*[H]\in\mathrm{Ker}(\Phi)$. This implies

$$\mathrm{Ker}(\Phi)=A(\overline{M(1)}).\{p([\omega])*[H],f_{2,3}([\omega])\}.$$

Moreover, by simple analysis of Zhu's algebra we get that

$$A(\boldsymbol{\mathcal{W}}_{2,3})=A_{-1}(\boldsymbol{\mathcal{W}}_{2,3})\oplus A_0(\boldsymbol{\mathcal{W}}_{2,3})\oplus A_1(\boldsymbol{\mathcal{W}}_{2,3})$$

where

$$A_0(\boldsymbol{\mathcal{W}}_{2,3}) = A(\overline{M(1)})/\mathrm{Ker}(\Phi), \quad \dim A_0(\boldsymbol{\mathcal{W}}_{2,3}) = 26,$$
$$A_1(\boldsymbol{\mathcal{W}}_{2,3}) = A_0(\boldsymbol{\mathcal{W}}_{2,3}).[E], \quad \dim A_1(\boldsymbol{\mathcal{W}}_{2,3}) = 6,$$
$$A_{-1}(\boldsymbol{\mathcal{W}}_{2,3}) = A_0(\boldsymbol{\mathcal{W}}_{2,3}).[F], \quad \dim A_{-1}(\boldsymbol{\mathcal{W}}_{2,3}) = 6.$$

More precisely we have that

$$p([\omega]) * [E] = p([\omega]) * [F] = 0,$$

$$A_1(\boldsymbol{\mathcal{W}}_{2,3}) = \mathrm{span}_{\mathbb{C}}\{p_h([\omega]) * [E], h \in S_{2,3}^{(2)}\},$$

$$A_{-1}(\boldsymbol{\mathcal{W}}_{2,3}) = \mathrm{span}_{\mathbb{C}}\{p_h([\omega]) * [F], h \in S_{2,3}^{(2)}\},$$

where

$$S_{2,3}^{(2)} = \{5, 7, 10/3, 33/8, 21/8, 35/24\}, \quad p(x) = (x - h)p_h(x).$$

This implies the following result:

Proposition 5.3. *We have*

$$\dim A(\boldsymbol{\mathcal{W}}_{2,3}) = 38.$$

Remark 4. *It is interesting to notice that in this case we most likely have*

$$\dim A(\boldsymbol{\mathcal{W}}_{2,3}) < \dim \mathcal{P}(\boldsymbol{\mathcal{W}}_{2,3}).$$

Remark 5. *In* [8]*, among other things, we classified irreducible $\boldsymbol{\mathcal{W}}_{2,p}$–modules. This can be now used to generalize results from this section for general p.*

6. Zhu's algebra $A(\mathcal{SW}(m))$

In this section we shall describe Zhu's algebra of the triplet vertex operator superalgebra $\mathcal{SW}(m)$ introduced in [6]. In [6] we classify irreducible $\mathcal{SW}(m)$–modules and proved that $\mathcal{SW}(m)$ is C_2–cofinite vertex operator superalgebra. There we also presented the conjecture that Zhu's algebra $A(\mathcal{SW}(m))$ is $(6m+1)$–dimensional. In this section we shall prove this conjecture by using new methods from previous sections.

We should say that $\mathcal{SW}(m)$ can be considered as a super-analog of triplet vertex algebra $\mathcal{W}(p)$, but $\mathcal{SW}(m)$ requires different techniques based on the representation theory of the $N=1$ Neveu-Schwarz Lie algebra.

We shall first recall definition of the triplet vertex superalgebra $\mathcal{SW}(m)$.

Let V_L be the lattice vertex superalgebra associated to the lattice $L = \mathbb{Z}\alpha$, with $\langle \alpha, \alpha \rangle = 2m+1$, $m \in \mathbb{N}$.

Let CL be the Clifford algebra, generated by $\{\phi(n), n \in \frac{1}{2} + \mathbb{Z}\} \cup \{1\}$ and relations

$$\{\phi(n), \phi(m)\} = \delta_{n,-m}, \quad n, m \in \tfrac{1}{2} + \mathbb{Z}.$$

Here 1 is central.

Let F be the CL–module generated by the vector $\mathbf{1}$ such that

$$\phi(n)\mathbf{1} = 0, \ n > 0.$$

Then the field
$$Y(\phi(-\tfrac{1}{2})\mathbf{1}, z) = \phi(z) = \sum_{n \in \frac{1}{2}+\mathbb{Z}} \phi(n) z^{-n-\frac{1}{2}},$$
generates the unique vertex operator superalgebra structure on F.

We define:
$$\begin{aligned}
\tau &= \frac{1}{\sqrt{2m+1}} \left(\alpha(-1)\mathbf{1} \otimes \phi(-\tfrac{1}{2})\mathbf{1} + 2m\mathbf{1} \otimes \phi(-\tfrac{3}{2})\mathbf{1} \right), \\
G(z) &= Y(\tau, z) = \sum_{n \in \mathbb{Z}} G(n + \tfrac{1}{2}) z^{-n-2}, \\
\omega &= \tfrac{1}{2} G(-\tfrac{1}{2})\tau, \quad L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}.
\end{aligned}$$

The components of the fields $L(z), G(z)$ define on $V_L \otimes F$ a representation of the $N=1$ Neveu-Schwarz superalgebra $\mathfrak{ns}$ with central charge $c_{2m+1,1} = \frac{3}{2}(1 - \frac{8m^2}{2m+1})$. Moreover, the operator
$$Q = \mathrm{Res}_z Y(e^{\alpha} \otimes \phi(-\tfrac{1}{2})\mathbf{1}, z)$$
is a screening operator which commutes with the action of the Neveu-Schwarz algebra.

The $N=1$ vertex operator superalgebra $\mathcal{SW}(m)$ is defined to be a subalgebra of $V_L \otimes F$ generated by superconformal vector τ and
$$F = e^{-\alpha}, \ H = Qe^{-\alpha}, \ E = Q^2 e^{-\alpha},$$
where these three vectors are highest weight vectors for the Neveu-Schwarz algebra $\mathfrak{ns}$.

For $X \in \{E, F, H\}$, we define $\widehat{X} = G(-1/2)X$.

The representation theory of $\mathcal{SW}(m)$ was developed in [6]. Recall that $\mathcal{SW}(m)$ is a simple, C_2–cofinite vertex superalgebra with $2m+1$ irreducible representations
$$S\Lambda(1), \dots, S\Lambda(m+1), \ S\Pi(1), \cdots, S\Pi(m).$$

For $i \in \mathbb{Z}$, we let
$$h^{2i+1,1} = \frac{(2m+1-2i-1)^2 - 4m^2}{8(2m+1)} = \frac{(m-i)^2 - m^2}{2(2m+1)}.$$
The top component of $S\Lambda(i+1)$ is 1-dimensional and has conformal weight $h^{2i+1,1}$, and the top component of $S\Pi(i)$ is 2–dimensional and has conformal weight $h^{2(3m+1-i)+1,1}$.

The vertex subalgebra of $\mathcal{SW}(m)$ generated by τ and H is called singlet vertex algebra and will be denoted by $\overline{SM(1)}$.

Theorem 6.1. [6] *Zhu's algebra $A(\overline{SM(1)})$ is isomorphic to the commutative algebra $\mathbb{C}[x,y]/\langle P(x,y)\rangle$, where $\langle P(x,y)\rangle$ is the principal ideal generated by*
$$P(x,y) = y^2 - C_m \prod_{i=0}^{2m} (x - h^{2i+1,1})$$
where $C_m = \frac{2^{2m+1}(2m+1)^{2m+1}}{(2m+1)!}$. (Here x and y correspond to $[\omega]$ and $[\widehat{H}]$.)

Recall that as a module over Neveu-Schwarz algebra $\mathcal{SW}(m)$ is generated by singular vectors

$$\{Q^j e^{-n\alpha},\ n \in \mathbb{Z}_{\geq 0}, 0 \leq j \leq 2n\},$$

and $\overline{SM(1)}$ is generated by singular vectors

$$\{Q^n e^{-n\alpha},\ n \in \mathbb{Z}_{\geq 0}\}.$$

For every $\ell \in \mathbb{Z}_{\geq 0}$, we define

$$\mathcal{SW}(m)_{-\ell} = \overline{SM(1)}.e^{-\ell\alpha} = \bigoplus_{n=0}^{\infty} U(\mathfrak{ns}).Q^n e^{-(n+\ell)\alpha},$$

$$\mathcal{SW}(m)_{\ell} = \overline{SM(1)}.Q^{2\ell} e^{-\ell\alpha} = \bigoplus_{n=0}^{\infty} U(\mathfrak{ns}).Q^{n+2\ell} e^{-(n+\ell)\alpha}.$$

Proposition 6.2. *For every $\ell \in \mathbb{Z}$, $\mathcal{SW}(m)_\ell$ is an irreducible $\overline{SM(1)}$–module and*

$$\mathcal{SW}(m) = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{SW}(m)_\ell.$$

Moreover, for $v \in \mathcal{SW}(m)_{\ell_1}$, $w \in \mathcal{SW}(m)_{\ell_2}$, we have

$$Y(v,z)w \in \mathcal{SW}(m)_{\ell_1+\ell_2}((z)).$$

Operator $G = Q$ satisfies conditions (5)-(9).

The proof of the following two lemmas is completely analogous as in the case of triplet vertex algebra $\mathcal{W}(p)$.

Lemma 6.3. *We have*

$$O(\mathcal{SW}(m))_0 \subset \bigoplus_{n=1}^{\infty} U(\mathfrak{ns}).Q^n e^{-n\alpha} + O(\overline{SM(1)}).$$

Define now two polynomials

$$\ell(x) = \prod_{i=2m+1}^{3m} (x - h^{2i+1,1}), \quad Sf_m(x) = \prod_{i=0}^{3m} (x - h^{2i+1,1}).$$

Lemma 6.4. *Assume that $n \geq 2$. Then in Zhu's algebra $A(\overline{SM(1)})$ we have*

$$[g\, Q^n e^{-n\alpha}] = A([\omega]) * [\widehat{H}] + B([\omega]) * Sf_m([\omega]),$$

where $A, B \in \mathbb{C}[x]$, $g \in U(\mathfrak{ns})$.

Now we shall consider the homomorphism

$$\Phi : A(\overline{SM(1)}) \to A(\mathcal{SW}(m))$$

from Section 3.

We need the following result:

Lemma 6.5. *Inside $A(\overline{SM(1)})$ we have*

$$[E \circ F] = D_m \ell([\omega]) * [\widehat{H}] = 0, \qquad D_m \neq 0.$$

Proof. We consider $[E \circ F]$ as an element of $A(\overline{SM(1)})$. Therefore

$$[E \circ F] = f([\omega]) * [\widehat{H}] + g([\omega])$$

for certain polynomials $f, g \in \mathbb{C}[x]$. Then we shall evaluate both sides of this equality on a family of $A(\overline{SM(1)})$–modules. Then Theorem 9.2 proven below (see also Proposition 8.1 and Appendix from [6]) gives

$$-\binom{2m}{m}^2 \binom{t+m}{4m+1} = f\left(\frac{t(t-2m)}{2(2m+1)}\right)\binom{t}{2m+1} + g\left(\frac{t(t-2m)}{2(2m+1)}\right)$$

for arbitrary $t \in \mathbb{C}$. This easily gives that $g(x) = 0$ and $f(x) = D_m \ell(x)$ for certain non-vanishing constant D_m. □

The previous result shows that

$$\ell([\omega]) * [\widehat{H}] \in \mathrm{Ker}(\Phi).$$

The proof of the following theorem is now completely analogous to that of Theorem 4.5.

Theorem 6.6. *We have*

$$\begin{aligned} \mathit{Ker}(\Phi) &= A(\overline{SM(1)}).(\ell([\omega]) * [\widehat{H}]) \\ &\cong \mathit{span}_{\mathbb{C}}\{A([\omega]) * \ell([\omega]) * [\widehat{H}] + B([\omega]) * Sf_m([\omega]), \quad A, B \in \mathbb{C}[x]\}. \end{aligned}$$

Let

$$A_0(\mathcal{SW}(m)) = \mathrm{Im}(\Phi) \cong A(\overline{SM(1)})/\mathrm{Ker}(\Phi).$$

By combining the results from Section 11 of [6] and the above results we obtain the description of Zhu's algebra $A(\mathcal{SW}(m))$. We get

$$A(\mathcal{SW}(m)) = A_{-1}(\mathcal{SW}(m)) \oplus A_0(\mathcal{SW}(m)) \oplus A_1(\mathcal{SW}(m)),$$

where

$$A_{-1}(\mathcal{SW}(m)) = A_0(\mathcal{SW}(m)).[\widehat{F}], \quad A_1(\mathcal{SW}(m)) = A_0(\mathcal{SW}(m)).[\widehat{E}],$$

and

$$\dim A_{\pm 1}(\mathcal{SW}(m)) = m, \quad \dim A_0(\mathcal{SW}(m)) \leq 4m+1.$$

More precisely, we have

$$\ell([\omega]) * [\widehat{E}] = \ell([\omega]) * [\widehat{F}] = 0,$$

$$A_1(\mathcal{SW}(m)) = \mathrm{span}_{\mathbb{C}}\{\ell_i([\omega]) * [\widehat{E}], \ \ 2m+1 \leq i \leq 3m+1\},$$

$$A_{-1}(\mathcal{SW}(m)) = \mathrm{span}_{\mathbb{C}}\{\ell_i([\omega]) * [\widehat{F}], \ \ 2m+1 \leq i \leq 3m+1\},$$

where $\ell(x) = \ell_i(x)(x - h^{2i+1,1})$.

Now Theorem 4.5 implies that $\dim A_0(\mathcal{W}(p)) = 4m+1$. Therefore $\dim A(\mathcal{SW}(m)) = 6m+1$. In this way we have proved the following theorem.

Theorem 6.7. *We have*

$$\dim A(\mathcal{SW}(m)) = 6m+1.$$

This theorem was conjectured in [6]. We now have completed description of the Zhu algebra $A(\mathcal{SW}(m))$.

Theorem 6.8. *Zhu's algebra $A(\mathcal{SW}(m))$ decomposes as a sum of ideals*

$$A(\mathcal{SW}(m)) = \bigoplus_{i=2m+1}^{3m} \mathbb{M}_{h^{2i+1,1}} \oplus \bigoplus_{i=0}^{m-1} \mathbb{I}_{h^{2i+1,1}} \oplus \mathbb{C}_{h^{2m+1,1}},$$

where $\mathbb{M}_{h^{2i+1,1}} \cong M_2(\mathbb{C})$, $\dim(\mathbb{I}_{h^{2i+1,1}}) = 2$ and $\mathbb{C}_{h^{2m+1,1}}$ is one-dimensional.

The ideals $\mathbb{I}_{h^{2i+1,1}}$ give a family of 2–dimensional indecomposable modules for $A(\mathcal{SW}(m))$. By applying Zhu's correspondence we get:

Corollary 6.9. *For every $1 \le i \le m$, there exits the logarithmic, self-dual, $\mathbb{Z}_{\ge 0}$–graded $\mathcal{SW}(m)$–module $\mathcal{SP}_i^+$ such that the top component $\mathcal{SP}_i^+(0)$ is two-dimensional and $L(0)$ acts on $\mathcal{SP}_i^+(0)$ (in some basis) as*

$$\begin{pmatrix} h^{2i+1,1} & 1 \\ 0 & h^{2i+1,1} \end{pmatrix}.$$

7. TWISTED ZHU'S ALGEBRA $A_\sigma(\mathcal{SW}(m))$

Every vertex operator superalgebra $V^{\bar 0} \oplus V^{\bar 1}$ has the canonical parity automorphism σ, where $\sigma_{V^{\bar 0}} = 1$ and $\sigma_{V^{\bar 1}} = -1$. We briefly recall the notion of σ-twisted Zhu's algebra (cf. [22]).

Consider the subspace $O_\sigma(V) \subset V$, spanned by elements of the form

$$\mathrm{Res}_x \frac{(1+x)^{\mathrm{wt}(u)}}{x^2} Y(u,x)v,$$

where $u \in V$ is homogeneous. It can be easily shown that

$$\mathrm{Res}_x \frac{(1+x)^{\mathrm{wt}(u)}}{x^n} Y(u,x)v \in O_\sigma(V) \text{ for } n \ge 2.$$

Then, the vector space $A_\sigma(V) = V/O_\sigma(V)$ is equipped with an associative algebra structure via

$$u * v = \mathrm{Res}_x \frac{(1+x)^{\mathrm{deg(u)}}}{x} Y(u,x)v$$

An important difference between the untwisted associative algebra and $A_\sigma(V)$ is that $A_\sigma(V)$ is $\mathbb{Z}_2$–graded, so

$$A_\sigma(V) = A_\sigma^0(V) \oplus A_\sigma^1(V).$$

We shall often use $[a] \in A_\sigma(V)$ for the image of $a \in V$ under the map $V \longrightarrow A_\sigma(V)$.

It is not hard to prove the following result.

Proposition 7.1. *Let V be a vertex operator superalgebra as in Section 2. There is a natural surjective superalgebra map from $\mathcal{P}(V)$ to $gr(A_\sigma(V))$.*

Here we shall consider the $N=1$ vertex operator superalgebras $\overline{SM(1)}$ and $\mathcal{SW}(m)$ and the corresponding twisted Zhu's algebras $A_\sigma(\overline{SM(1)})$ and $A_\sigma(\mathcal{SW}(m))$.

Theorem 7.2. [5] *The associative algebra $A_\sigma(\overline{SM(1)})$ is isomorphic to the $\mathbb{Z}_2$–graded commutative associative algebra*

$$\mathbb{C}[x,y]/\langle H(x,y)\rangle$$

where $\langle H(x,y)\rangle$ is (two-sided) ideal in $\mathbb{C}[x,y]$, generated by

$$H(x,y) = y^2 - \widetilde{C}_m \prod_{i=0}^{m-1}\left(x^2 - \frac{(2i+1-2m)^2}{8(2m+1)}\right)^2,$$

where $\widetilde{C}_m = \frac{2^{2m-1}(2m+1)^{2m}}{(2m)!^2}$. (Here x and y correspond to $[\tau]$ and $[H]$).

Define

$$h^{2i+2,1} = \frac{(2m+1-2i-2)^2 - 4m^2}{8(2m+1)} + \frac{1}{16}.$$

We also have a natural homomorphism $\Phi : A_\sigma(\overline{SM(1)}) \to A_\sigma(\mathcal{SW}(m))$. By using similar approach as in the case of untwisted Zhu's algebras $A(\mathcal{W}(p))$ and $A(\mathcal{SW}(m))$ we get the following result.

Proposition 7.3. *Ker(Φ) is contained in the following ideal in $A_\sigma(\overline{SM(1)})$*

$$A_\sigma(\overline{SM(1)}).\{r([\omega]) * [H]\} = \{A([\tau]) * r([\omega]) * [H] + B([\tau]) * Rf_m([\omega]),\ \ A, B \in \mathbb{C}[x]\},$$

where

$$\begin{aligned} [\omega] = &\quad [\tau]^2 + \frac{c_{2m+1,1}}{24}, \\ r(x) = &\quad \prod_{i=2m}^{3m}(x - h^{2i+2,1}), \\ Rf_m(x) = &\quad \prod_{i=0}^{3m}(x - h^{2i+2,1}). \end{aligned}$$

Corollary 7.4. *The center of Zhu's algebra $A_\sigma(\mathcal{SW}(m))$ is $6m+2$–dimensional and it is isomorphic to*

$$\mathbb{C}[x]/\langle Rf_m(x^2 + \tfrac{c_{2m+1,1}}{24})\rangle.$$

Define the following two vectors in $\overline{SM(1)}$:

$$U^{F,\widehat{E}} = \mathrm{Res}_z Y(F,z)\widehat{E}\frac{(1+z)^{2m+\frac{1}{2}}}{z^2}, \quad U^{F,E} = \mathrm{Res}_z Y(F,z)E\frac{(1+z)^{2m+\frac{1}{2}}}{z^3}.$$

Lemma 7.5. *In $A_\sigma(\overline{SM(1)})$ we have*

$$[U^{F,\widehat{E}}] = A_m\, r([\omega]) * [H], \quad A_m \neq 0.$$

Proof. Let v_λ be the highest weight vector in σ–twisted $\overline{SM(1)}$–module $M(1,\lambda)\otimes M$, where $M(1,\lambda)$ is an irreducible module for the Heisenberg vertex algebra $M(1)$ and M is the σ–twisted module for Clifford vertex superalgebra F (see [5] for details). By direct calculation which uses the concepts from [5] we get

$$o(U^{F,\widehat{E}})v_\lambda = (R_m^{(1)}(t) + R_m^{(2)}(t))v_\lambda$$

where

$$\begin{aligned} R^{(1)}_m(t) &= \nu_1 Res_{z_1,z_2,z_3} \frac{(1+z_1)^{2m+1/2-t}}{z_1^2}(z_1z_2z_3)^{-2m-1} \\ &\cdot(1-z_2/z_1)^{-2m-1}(1-z_3/z_1)^{-2m-1}(z_2-z_3)^{2m}(1+z_2)^{t+1/2}(1+z_3)^{t-1/2}, \end{aligned}$$

$$\begin{aligned} R^{(2)}_m(t) &= \nu_2 Res_{z_1,z_2,z_3} \frac{(1+z_1)^{2m+1/2-t}}{z_1}(z_1z_2z_3)^{-2m-2} \\ &\cdot(1-z_2/z_1)^{-2m-1}(1-z_3/z_1)^{-2m-1}(z_2-z_3)^{2m+2}(1+z_2)^{t-1/2}(1+z_3)^{t-1/2} \end{aligned}$$

and ν_1, ν_2 are non-zero complex numbers. Now Theorem 9.4 gives that

$$R^{(1)}_m(t) = 0, \qquad R^{(2)}_m(t) = \nu \binom{t+m+1/2}{4m+2} \qquad (\nu \neq 0).$$

This easily implies that in the twisted Zhu's algebra $A_\sigma(\overline{SM(1)})$ we have

$$[U^{F,\widehat{E}}] = A_m r([\omega]) * [H], \quad (A_m \neq 0).$$

The proof follows. □

Lemma 7.6. *Inside $A_\sigma(\overline{SM(1)})$ we have*

$$[U^{F,E}] = B_m \, r([\omega]) * [\tau] * [H], \quad B_m \neq 0.$$

Proof. Everything here is a matter of rewriting the residue explicitly. The rest follows from Theorem 9.3. □

By using Lemma 7.5 (or using Lemma 7.6 and the fact that $[\tau]$ is a unit in $A_\sigma(\mathcal{SW}(m))$), we have

$$r([\omega]) * [H] \in \mathrm{Ker}(\Phi).$$

Therefore,

$$\mathrm{Ker}(\Phi) = A_\sigma(\overline{SM(1)}).\{r([\omega]) * [H]\}. \tag{17}$$

Now we are in position to describe Zhu's algebra of $A_\sigma(\mathcal{SW}(m))$. By using Proposition 6.2 of [5] we get:

$$A_\sigma(\mathcal{SW}(m)) = A_\sigma(\mathcal{SW}(m))_{-1} \oplus A_\sigma(\mathcal{SW}(m))_0 \oplus A_\sigma(\mathcal{SW}(m))_1,$$

where

$$A_\sigma(\mathcal{SW}(m))_0 = A_\sigma(\overline{SM(1)})/\mathrm{Ker}(\Phi),$$
$$A_\sigma(\mathcal{SW}(m))_{-1} = A_\sigma(\mathcal{SW}(m))_0.[F], \quad A_\sigma(\mathcal{SW}(m))_1 = A_\sigma(\mathcal{SW}(m))_0.[E].$$

Relation (17) gives that $\dim A_\sigma(\mathcal{SW}(m))_0 = 8m+4$. One can also see that

$$A_\sigma(\mathcal{SW}(m))_1 = \mathrm{span}_{\mathbb{C}}\{r_i^\varepsilon([\tau]) * [E],\ 2m \leq i \leq 3m,\ \varepsilon = \pm\},$$
$$A_\sigma(\mathcal{SW}(m))_{-1} = \mathrm{span}_{\mathbb{C}}\{r_i^\varepsilon([\tau]) * [F],\ 2m \leq i \leq 3m,\ \varepsilon = \pm\},$$

where

$$r(x^2 + \tfrac{c_{2m+1,1}}{24}) = r_i^{\pm}(x)(x \pm \frac{2m-2i-1}{\sqrt{8(2m+1)}}).$$

Therefore:

$$\dim A_\sigma(\mathcal{SW}(m))_{\pm 1} = 2m + 2.$$

In this way we have proved the following result (conjectured in [5]):

Theorem 7.7. *We have*

$$\dim A_\sigma(\mathcal{SW}(m)) = 12m + 8.$$

8. The C_2–algebra $\mathcal{P}(\mathcal{SW}(m))$

Now we are in the position to determine $\mathcal{P}(\mathcal{SW}(m))$. Firstly, observe that

$$\dim \mathcal{P}(\mathcal{SW}(m)) \geq \dim A_\sigma(\mathcal{SW}(m)) = 12m + 8.$$

Therefore we only have to prove that

$$\dim \mathcal{P}(\mathcal{SW}(m)) \leq 12m + 8.$$

By using results from [6] we conclude that $\mathcal{P}(\mathcal{SW}(m))$ is generated by

$$\overline{\tau}, \overline{\omega}, \overline{E}, \overline{F}, \overline{H}, \overline{\widehat{E}}, \overline{\widehat{F}}, \overline{\widehat{H}}.$$

Also the following relations hold:

$$\overline{\tau}^2 = \overline{\omega}^{3m+1} = \overline{\widehat{E}}^2 = \overline{\widehat{F}}^2 = \overline{\widehat{H}} \cdot \overline{\widehat{E}} = \overline{\widehat{H}} \cdot \overline{\widehat{F}} = 0,$$

$$\overline{X}^2 = \overline{\tau}\overline{\widehat{X}} = 0, \overline{\tau}\overline{X} = \nu_1 \overline{\widehat{X}},\ \overline{\widehat{H}}^2 = \nu_2 \overline{\omega}^{2m+1},$$

where ν_1, ν_2 are non-zero complex numbers and $X \in \{E, F, H\}$.

Therefore every element $u \in \mathcal{P}(\mathcal{SW}(m))$ has the form

$$u = \quad f_1(\overline{\omega}) + f_2(\overline{\omega})\overline{E} + f_3(\overline{\omega})\overline{F} + f_4(\overline{\omega})\overline{H} + g_1(\overline{\omega})\overline{\tau} + g_2(\overline{\omega})\overline{\widehat{E}} + g_3(\overline{\omega})\overline{\widehat{F}} + g_4(\overline{\omega})\overline{\widehat{H}},$$

for certain polynomials $f_i, g_i \in \mathbb{C}[x],\ \deg(f_i), \deg(g_i) \leq 3m,\ i = 1, \cdots, 4.$

By using Lemma 7.5 and Lemma 7.6 we get:

Proposition 8.1. *We have*

$$F_{-2}\widehat{E} \equiv A_m L(-2)^{m+1} H + v_1 \quad mod(C_2(\mathcal{SW}(m))) \qquad (A_m \neq 0), \tag{18}$$

$$F_{-3}E \equiv B_m L(-2)^{m+1} \widehat{H} + v_2 \quad mod(C_2(\mathcal{SW}(m))) \qquad (B_m \neq 0), \tag{19}$$

where $v_1, v_2 \in U(\mathfrak{ns}).\mathbf{1}$, $\mathrm{wt}(v_1) = 4m + 5/2$, $\mathrm{wt}(v_2) = 4m + 3$.

Proof. Let us prove relation (18). First we noticed $F_{-2}\widehat{E}$ is an odd vector in $\overline{SM(1)}$ of conformal weight $4m + 5/2$. This easily implies that $F_{-2}\widehat{E}$ has the form

$$F_{-2}\widehat{E} = AL(-2)^{m+1}H + v'$$

for certain $A \in \mathbb{C}$ and $v' \in C_2(\mathcal{SW}(m)))$. Then relation in Zhu's algebra $A_\sigma(\overline{SM(1)})$ from Lemma 7.5 gives that $A = A_m \neq 0$.

Relation (19) follows from Lemma 7.6 in the same way. $\square$

By using Proposition 8.1 and the action of the operator Q we get:

$$\overline{\omega}^{m+1}\overline{X} = \overline{\omega}^{m+1}\overline{\widehat{X}} = 0 \qquad X \in \{E, F, H\},$$

The analysis above implies that $\dim \mathcal{P}(\mathcal{SW}(m)) \leq 12m + 8$. In this way we have proved the following result:

Theorem 8.2. *We have*

$$\dim \mathcal{P}(\mathcal{SW}(m)) = \dim A_\sigma(\mathcal{SW}(m)).$$

From the description of $\mathcal{P}(\mathcal{SW}(m))$ we see $\dim \mathcal{P}_0(\mathcal{SW}(m)) = (3m+1) + 3(m+1) = 6m+4$. Consequently:

Corollary 8.3. *For all $m \in \mathbb{N}$,*

$$\dim A(\mathcal{SW}(m)) < \dim \mathcal{P}_0(\mathcal{SW}(m)). \tag{20}$$

9. Constant term identities

In this section, which is mostly of combinatorial nature, we obtain constant (or residue) term identities needed in the paper. We are interested in certain multiple sums, which after several steps reduce to a single sum. Although we only need non-vanishing condition for these sums, we in fact provide closed evaluation expression which is of independent interest. The main tool is Wilf-Zeilberger (WZ) theory of summation elaborated in more details in the appendix.

As usual, for $t \in \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 0}$ we set

$$\binom{t}{k} = \frac{t(t-1)\cdots(t-k+1)}{k!}.$$

Define also

$$H_p(t) = \binom{2p}{p}\binom{2p-2}{p-1}\binom{t+p}{4p-1}.$$

Theorem 9.1. *Let*

$$G_p(t) = o(E \circ F)v_\lambda = \sum_{i\geq 0}\binom{2p-1}{i} o(E_{-2+i}F)v_\lambda = G_p(t)v_\lambda, \tag{21}$$

where we view t as a formal variable. Then

$$G_p(t) = \mathrm{Res}_{x_1,x_2,x_3}\frac{1}{(x_1x_2x_3)^{2p}}\frac{(x_2-x_3)^{2p}}{(1-x_2/x_1)^{2p}(1-x_3/x_1)^{2p}}\frac{(1+x_1)^{2p-1-t}}{x_1^2}(1+x_2)^t(1+x_3)^t,$$

and

$$G_p(t) = H_p(t). \tag{22}$$

Proof.

Easy inspections shows that $G_p(t)$ is a polynomial in t.

Formula (21) has already been proven in [4], so we only focus on (22). The proof is divided into 5 steps:

In Step 1 we prove $G_p(t)$ vanishes at $t \in \{0, \ldots, 2p-1\}$.

In Step 2 we prove $G(t) = (-1)^p G(2p-2-t)$ (the skew-symmetry).

In Step 3 we prove $G_p(t)$ vanishes at $t \in \{-p, \ldots, -1\}$.

In Step 4 we apply skew-symmetry to show that $G_p(t)$ vanishes at $t \in \{3p-2, \ldots, 2p\}$. Steps 1-4 imply that $G_p(t)$ is divisible by $\binom{t+p}{4p-1}$.

In Step 5 we show that

$$H'(0) = G'(0) \neq 0,$$

that is $H(t)$ and $G(t)$ have the same first derivative at zero.

Step 1. This is an easy observation, which follows simply from consideration of $Res_{x_1} G_p(t)$. For every $t = i \in \{1, ..., 2p-1\}$ the highest positive powers of x_1 appearing in the Laurent expansion of $G_p(t)$ is at most $2p-1-i$, coming from the expansion of $(1+x_1)^{2p-1-i}$. Contribution from other terms containing x_1 is $x_1^{-2p-2-j}$, where $j \geq 0$. Thus $G_p(i) = 0$. If $t = i = 0$ one easily sees that the constant term is zero.

Step 2. Apply the substitution $y_i = \frac{x_i}{1+x_i}$ and the general formula

$$\mathrm{Res}_{y_i} F(y_i) = \mathrm{Res}_{x_i} y'(x_i) F(y(x_i)),$$

where $y_i = x_i + \cdots \in x_i \mathbb{C}[[x_i]]$.

Step 3. This part is more tricky because for $t = -1, -2, .., -p$ the terms $(1+x_i)^t$ have an infinite power expansion in x_i. So let $-t = k \in \{1, ..., p\}$. We clearly have

$$\frac{1}{(x_1x_2x_3)^{2p}} \frac{((1+x_2)-(1+x_3))^{2p}}{(1-x_2/x_1)^{2p}(1-x_3/x_1)^{2p}} \frac{(1+x_1)^{2p-1+k}}{x_1^2} (1+x_2)^{-k}(1+x_3)^{-k},$$

(23)

$$= \sum_{i+j=2p} (-1)^j \binom{2p}{i} \frac{1}{(x_1x_2x_3)^{2p}(1-x_2/x_1)^{2p}(1-x_3/x_1)^{2p}} \frac{(1+x_1)^{2p-1-t}}{x_1^2} \frac{(1+x_2)^i}{(1+x_2)^k} \frac{(1+x_2)^j}{(1+x_3)^k},$$

Now, for every k in the range we either have

$$i \geq k, \quad \text{or} \quad j \geq k,$$

(otherwise $2p = i + j < 2k$, contradicting our choice of k). If $i \geq k$ we will consider Res_{x_2} of (23) (if $j \geq k$ we consider Res_{x_3} instead and the argument follows verbatim). In the expression $(1+x_2)^{i-k}$ the highest power of x_2 is clearly $i-k$. In addition we already have x_2^{-2p} contribution, so the highest power of x_2 we get from these two terms is $x_2^{-2p+i-k}$. For Res_{x_2} to be nontrivial we must have additional $(2p+k-i-1)$ powers of x_2. This can come only from the expansion of $(1-x_2/x_1)^{-2p}$, meaning that we need term $(x_2/x_1)^{2p+k-i-1}$ in its expansion (and higher powers). Now, we consider Res_{x_1}. We already have in (23) the factor $\frac{1}{x_2^{2p+2}}$, so we have $\frac{1}{x_1^{4p+2+k-i-1}}$ as the term with the highest power of x_1. The highest positive power of x_1 is $(2p-1+k)$, which comes from expansion of $(1+x_1)^{2p-1+k}$. Since we are taking Res_{x_1} we see that

$$2p-1+k-(4p+1+k-i) = -2p-2+i,$$

which is always ≤ -2, thus gives the zero contribution to Res_{x_1}, and $G_p(k) = 0$.

Step 4. Since $G_p(t) = (-1)^p G_p(2p-2-t)$, Step 3 gives $G_p(i) = 0$, for $i = 2p, ..., 3p-2$.

Step 5. Here, it is convenient to rewrite $G_p(t)$ as

$$\sum_{(i,j,k)=(1,0,0)}^{(2p-1,2p-i-1,i-1)} \binom{2p}{i}\binom{-2p}{j}\binom{-2p}{k}\binom{t}{2p-1-i-j}\binom{t}{i-1-k}\binom{2p-1-t}{2p+j+k+1}. \tag{24}$$

We have to determine the linear coefficients in $G_p(t)$ (as we already know the constant term is zero). Observe that

$$\binom{2p-1-t}{2p+j+k+1} \in \lambda_1 t + \cdots \in t\mathbb{C}[t],$$

where $\lambda_1 \neq 0$ for all j and k! Similarly, for $2p-1-i-j \neq 0$ and $i-1-k \neq 0$ we also have

$$\binom{t}{2p-i-j-1} \in \nu_1 t + \cdots, \ \ \nu_1 \neq 0$$

and

$$\binom{t}{i-1-k} \in \epsilon_1 t + \cdots, \ \ \epsilon_1 \neq 0,$$

and trivially $\binom{t}{0} = 1$, if $2p-1-i-j = 0$ or $i-1-k = 0$. In conclusion, to extract the linear term from $G_p(t)$, it is sufficient to consider the case

$$2p-1-i-j = 0, \ \ i-1-k = 0. \tag{25}$$

With this choice

$$\binom{2p-1-t}{2p+j+k+1} = \frac{(2p-1)!(2p-1)!}{(4p-1)!}t + \cdots,$$

where dots denote the higher powers of t. For j and k subject to (25), we have $j = 2p-i-1$ and $k = i-1$, so we now have

$$G_p(t) = \frac{(2p-1)!^2}{(4p-1)!}t\left(\sum_{i=1}^{2p-1}(-1)^i\binom{2p}{i}\binom{-2p}{2p-1-i}\binom{-2p}{i-1}\right) + \cdots,$$

where again the dots denote the higher order terms. The sum in the parenthesis, denoted by $f(p)$, can be evaluated via Zeilberger's algorithm (cf. Appendix). We get

$$3(3p-4)(2p-3)(3p-2)f(p-1) + f(p)(2p-1)(p-1)^2 = 0, \ \ p \geq 2$$

The last formula yields (after iteration)

$$f(p) = (-1)^p \frac{2(3p-2)!}{(2p-1)(p-1)!^3},$$

so we finally have

$$G_p(t) = (-1)^p \frac{(2p-1)!^2}{(4p-1)!}\frac{2(3p-2)!}{(2p-1)(p-1)!^3}t + \cdots = (-1)^p \frac{(2p)!(2p-2)!(3p-2)!}{p!(p-1)!^2(4p-1)!}t + \cdots,$$

But this coefficient is precisely the linear coefficients of

$$H_p(t) = (-1)^p \binom{2p}{p}\binom{2p-2}{p-1}\frac{(3p-2)!p!}{(4p-1)!}t + \cdots ,$$

and $H'(0) = G'(0)$.

$\square$

Remark 6. *1. It is tempting to ask whether WZ-theory can be applied directly to (24). The short answer seems to be "no", or at least we couldn't make it work even after we simplify (24) to a single sum involving generalized hypergometric series.*

2. There seems to be another degree of freedom in the formula for $f(p)$. We can for instance show that for $k \geq 1$, $p \geq k$:

$$\sum_{i=k}^{2p-k}(-1)^i\binom{2p}{i}\binom{-2p}{2p-k-i}\binom{-2p}{i-k} = \frac{2(-1)^p(3p-1-k)!}{(2p-k)(p-1)!^2(p-k)!}.$$

Our formula for $f(p)$ is obtained by specializing $k=1$.

Theorem 9.2. *The residue*

$$\sum_{i=0}^{2m}\mathrm{Res}_{z_1,z_2,z_3}\frac{(1+z_1)^{2m}}{z_1}z_2^{-i-1}z_3^i(z_1z_2z_3)^{-2m-1}$$

$$\cdot(1-z_2/z_1)^{-2m-1}(1-z_3/z_1)^{-2m-1}(z_2-z_3)^{2m+1}\frac{(1+z_2)^t(1+z_3)^t}{(1+z_1)^t}$$

equals

$$-\binom{2m}{m}^2\binom{t+m}{4m+1}.$$

Proof. The proof is analogous to the previous theorem so we omit some details. Let $\tilde{H}_m(t) = -\binom{2m}{m}^2\binom{t+m}{4m+1}$. Observe first that the sum in question can be rewritten more compactly as

$$\mathrm{Res}_{z_1,z_2,z_3}\frac{(1+z_1)^{2m}}{z_1}(z_1z_2z_3)^{-2m-1}.$$

$$\cdot(1-z_2/z_1)^{-2m-1}(1-z_3/z_1)^{-2m-1}(z_2-z_3)^{2m}\frac{(1+z_2)^t(1+z_3)^t}{(1+z_1)^t},$$

which equals

(26)

$$\tilde{G}_m(t) := \sum_{(i,j,k)=(0,0,0)}^{(2m,2m-i,i)}(-1)^{i+j+k}\binom{-2m-1}{k}\binom{-2m-1}{j}\binom{2m}{i}\binom{t}{i-k}\binom{t}{2m-j-i}\binom{2m-t}{2m+1+k+j}.$$

As in the previous theorem we analyze the roots of $\tilde{G}_m(t)$ and show it is divisible by $\binom{t+m}{4m+1}$. Then we write

$$\tilde{H}_m(t) = -\frac{(2m)!^2}{(4m+1)!}t\left(\sum_{i=0}^{2m}(-1)^i\binom{-2m-1}{i}\binom{-2m-1}{2m-i}\binom{2m}{i} + \cdots\right).$$

To prove

$$\frac{(2m)!^2}{(4m+1)!}\sum_{i=0}^{2m}(-1)^i\binom{-2m-1}{i}\binom{-2m-1}{2m-i}\binom{2m}{i}=\binom{2m}{m}^2\frac{m!(-1)^m(3m)!}{(4m+1)!}. \tag{27}$$

we denote the sum on the left hand side in (27) by $Sum(m)$. By using Zeilberger's algorithm we obtain

$$-Sum(m+1)(m+1)^2-3(3m+1)(3m+2)Sum(m)=0,$$

so we get

$$Sum(m)=\sum_{i=0}^{2m}(-1)^i\binom{-2m-1}{i}\binom{-2m-1}{2m-i}\binom{2m}{i}=\frac{(-1)^m(3m)!}{m!^3}, \tag{28}$$

which yields the claim (see Appendix for more details). To finish the proof we only have to argue

$$\tilde{H}'_m(0)=\tilde{G}'_m(0),$$

which is easy to check. □

Theorem 9.3. *For $m\in\mathbb{N}$, we have*

$$\mathrm{Res}_{z_1,z_2,z_3}\frac{(1+z_1)^{2m+1/2-t}}{z_1^3}(z_1z_2z_3)^{-2m-1}$$

$$\cdot(1-z_2/z_1)^{-2m-1}(1-z_3/z_1)^{-2m-1}(z_2-z_3)^{2m}(1+z_2)^{t+1/2}(1+z_3)^{t-1/2}$$

equals

$$\frac{1}{(4m+3)(2m-1)}\binom{2m}{m}\binom{2m+1}{m}(t-m)\binom{t+\frac{1}{2}+m}{4m+2}.$$

Proof. Compared to the proof of Theorem 9.2 the strategy here is a bit different. We denote the residue by $F(m,t)$ and the product of binomial coefficients by $G(m,t)$.

As in Step 1 of Theorem 9.1 we first identify trivial half-integer zeros. Then (as in Step 2) we obtain the symmetry identity

$$F(m,t)=-F(m,2m-t).$$

The last relation also implies $F(m,m)=0$. Another application of the same formula gives the remaining zeros.

In the last step we shall argue $G'(m,t_0)=F'(m,t_0)$ for some $t=t_0$. It turns out that $G'(m,0)$ is hard to analyze so we choose $t_0=\frac{1}{2}$ instead. Then again, as in Theorem 9.1, we rewrite first

$$F(m,t)=\sum_{j,k,i}(-1)^{j+k+i}\binom{2m+\frac{1}{2}-t}{2m+3+j+k}\binom{-2m-1}{j}\binom{-2m-1}{k}\binom{2m}{i}\binom{t+\frac{1}{2}}{i-k}\binom{t-\frac{1}{2}}{2m-j-i}.$$

To compute $F'(m,t)|_{t=1/2}$ we have to analyze two sums:

$$Sum(m)=\sum_{i=0}^{2m}(-1)^i\binom{-2m-1}{2m-i}\binom{-2m-1}{i}\binom{2m}{i}. \tag{29}$$

This sum was already computed in (28). We also need

$$TSum(m) = \sum_{i=0}^{2m-1} (-1)^i \binom{-2m-1}{2m-i-1} \binom{-2m-1}{i} \binom{2m}{i}. \tag{30}$$

It is not hard to see by using Zeilberger's algorithm that $TSum(m) = -\frac{1}{2}Sum(m)$. Putting everything together we get a closed expression for $F'(m,t)|_{t=1/2}$. It is trivial to see that it also equals $G'(m,t)|_{t=1/2}$. The proof follows. □

By using similar methods we can also prove the following result.

Theorem 9.4. *For $m \in \mathbb{N}$, we have*

(i)

$$\begin{aligned} &\mathrm{Res}_{z_1,z_2,z_3} \frac{(1+z_1)^{2m+1/2-t}}{z_1^2} (z_1 z_2 z_3)^{-2m-1} \\ &\cdot (1-z_2/z_1)^{-2m-1} (1-z_3/z_1)^{-2m-1} (z_2-z_3)^{2m} (1+z_2)^{t+1/2} (1+z_3)^{t-1/2} = 0. \end{aligned} \tag{31}$$

(ii)

$$\begin{aligned} &\mathrm{Res}_{z_1,z_2,z_3} \frac{(1+z_1)^{2m+1/2-t}}{z_1} (z_1 z_2 z_3)^{-2m-2} \\ &\cdot (1-z_2/z_1)^{-2m-1} (1-z_3/z_1)^{-2m-1} (z_2-z_3)^{2m+2} (1+z_2)^{t-1/2} (1+z_3)^{t-1/2} \\ = \ & -\binom{2m}{m}^2 \frac{2m+1}{m+1} \binom{t+m+1/2}{4m+2}. \end{aligned}$$

10. Appendix

Let us briefly outline Zeilberger algorithm method following the book [21]. One is generally interested in closed expression for finite sum

$$f(n) = \sum_i F(n,i),$$

where $F(n,i)$ is hypergeometric in both arguments (meaning that $F(n,i+1)/F(n,i)$ and $F(n+1,i)/F(n,i)$ are rational functions). The main idea behind Zeilberger algorithm, also known as the method of *creative telescoping*, is to find another function $R(n,i)$ and the following recurrence:

$$\sum_{j=0}^{k} a_j(n) F(n+j,i) = R(n,i+1) - R(n,i),$$

where $a_j(n)$ are polynomials in n. Assume for a moment that we are able to find such $R(n,i)$, which is nonzero for finitely many i. Then summing over $i \in \mathbb{Z}$, yields the recursion

$$\sum_{j=0}^{k} a_j(n) f(n+j) = 0.$$

If there is such $R(n,i)$ Zeilberger's algorithm can find it, and this part is implemented in various Maple/Mathematica packages (e.g. `sumtools`). We should say that in all our applications $k=1$ (first order recursions), which can be easily solved and we get closed expression for $f(n)$.

We illustrate the method on the identity (27), or equivalently (28). Let

$$F(m,i) = (-1)^i \binom{-2m-1}{i}\binom{-2m-1}{2m-i}\binom{2m}{i}.$$

Zeilberger's algorithm gives

$$R(m,i) = G(m,i)F(m,i),$$

$$G(m,i) = \frac{i^2(-i+4m+1)\Big((m+1)i^4 - 2(m+1)(5m+4)i^3 + (m+1)(120m^2+168m+59)i^2}{4(m+1)(2m+1)(-i+2m+1)^2(-i+2m+2)^2}$$

$$\frac{-2(m+1)(260m^3+558m^2+395m+92)i + 4(m+1)(172m^4+506m^3+548m^2+259m+45)\Big)}{4(m+1)(2m+1)(-i+2m+1)^2(-i+2m+2)^2}.$$

It is easy to prove the identity

$$-(m+1)^2F(m+1,i) - 3(3m+1)(3m+2)F(m,i) = R(m,i+1) - R(m,i). \tag{32}$$

We shall also need

$$\begin{aligned}(m+1)^2(F(m+1,2m) + F(m+1,2m+1) + F(m+1,2m+2))\\ +3(3m+1)(3m+2)F(m,2m) = R(m,2m).\end{aligned} \tag{33}$$

Now we sum the equation (32) over $i \in \{0,..,2m-1\}$ and use (33). We obtain

$$-(m+1)^2f(m+1) - 3(3m+1)(3m+2)f(m) = 0$$

as desired. This immediately gives

$$f(m) = \sum_{i=0}^{2m}(-1)^i\binom{-2m-1}{i}\binom{-2m-1}{2m-i}\binom{2m}{i} = \frac{(-1)^m(3m)!}{m!^3}.$$

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